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XII.

ON THE METHOD OF LEAST SQUARES.

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The method of least squares with the theory of errors of observation upon which it depends forms the foundation of modern practical astronomy, and is largely used in subjects akin to astronomy, as geodesy and physics: the object of the present note is to assist in their practical application, both in planning and reducing series of observations.

Legendre first seems to have published the method of least squares, mainly as a convenient and pretty method of computation. Gauss soon after showed that it was the same in principle as the ordinary method of taking the arithmetical means, in the simpler cases to which that is applicable; and that either method presupposes a distribution of the errors of observation, such that the probability of larger errors is less than that of smaller; and proportional to the function

$$e^{-h^2\Delta^2}$$

where Δ is an error of observation, and h a factor which reduces all systems of such errors to be comparable with each other.

Of course the probability of an error of observation exactly equal to a given amount, no hair more nor less, is infinitesimal; and the definite integral

$$\frac{h}{\sqrt{\pi}}\int_{\Delta_1}^{\Delta_2} e^{-h^2\Delta^2} d\Delta$$

denotes the probability that there will be errors between the limits Δ_1 and Δ_2; while the total probability of all the errors of observation denoted by unity will be equal to

$$\frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-h^2\Delta^2} d\Delta.$$

Shortly after this investigation of Gauss, Bessel examined long series of observations, to see whether the law of distribution of errors thus indicated was a true one: he found that it was approximately so. His tables and some results are in the Fundamenta Astronomiæ: from these it appears that the definite integral above mentioned does represent the actual distribution of errors with striking exactness; but that there is generally a surplus of perhaps 1 to 3 per cent of the larger errors.

I may here mention that the least favorable series quoted by Bessel — Bradley's declinations — are now in process of re-reduction by Prof. Auwers, of Berlin, and that his results in part are in my hands for another purpose. The larger discrepancies which Bessel's own reduction left in them will probably be found to disappear in the newer calculations, and seem to arise from variations in the zero point of the quadrant.

A few years after Bessel's results were published, Gauss wrote his Theoria Combinationis Observationum. In this he takes the ground, from the beginning, that

$$e^{-h^2 \Delta^2}$$

does represent the probability of error, and mentions casually that it is only an approximation.

About 1838, Bessel published his last paper upon this subject. It seems to be little known in this country, but is extremely important.

He shows that the law of error will be that mentioned with greater approximation, the nearer the following conditions are complied with.

First, that the sources of error are very numerous.

Second, that they give rise to errors of equal average magnitude.

He then points out that the first condition always holds good, by an enumeration of the known sources of error; and that in good observations the second condition has always a tendency to maintain itself, because if any one source of error is sensibly more influential than the rest, it will be detected and put away, or at least its effect diminished by a proper arrangement of the work.

The main object of this paper is to give the rules for good observing derived from this theory: I have tested them in two long series of observations, one made at Cambridge from 1862 to 1866, the other at Chicago from 1868 to 1871. The first series is of right-ascensions of the principal stars, about 500 in number, each observed at least eight times, and many from thirty to forty times. The second series is a portion of the great zone observations now going on under the charge of the Astronomische Gesellschaft of Leipzig. The rules for this series were formulated by Argelander.

The sources of error (not mistakes) in astronomical observations are partly psychological (deficiency of attention), partly psycho-physical (dulness of the senses, time expended in communications through the nerves), partly instrumental, or depending on temperature, partly optical, depending on the condition of the atmosphere. The first rule then is that the observer must keep himself in uniform condition, and therefore be temperate and regular in his life. He must keep his senses constantly under control. He must have good instruments, well and firmly mounted: all the parts of the instruments must be solid. The different instruments must correspond with each other in their degree of perfection, and must always be in good repair. Observations must not be made when the air is uncommonly disturbed, or when the observer cannot keep himself warm enough to be comfortable, — of course when it is practicable to make the observations at all under better circumstances.

The single observations should be uniform; *i. e.*, on nearly the same number of wires, with nearly the same number of settings and microscope readings.

Too great fatigue should be avoided by timely pauses, so that, for instance, the first observations of a night may not be very good, and the last very bad.

Long series of observations have been greatly damaged by the following causes, among others: —

Large errors of division, much exceeding the errors of setting upon a star.

The wearing at the centre of a quadrant; and a gradual flexure of the whole instrument.

Placing a transit instrument in a high tower, which expanded and contracted by the sun's heat. Too great trust in the fixity of an instrument. Closely counterpoising a meridian circle, so that, when observing zones in a hurry, the axis was lifted out of its bearings.

Weakness of the telescope tube, and an attempt to improve it by levers of flexure.

The wearing of the pivots of a transit instrument. Employing a person to note time for the observer proper.

Negligence in determining the zero-points ; too great trust in the Nautical Almanac in comparison with immediate observation.

The employment of two observers of very unequal skill upon the same work. The employment of a careless or ignorant person to direct good observers.

In all these cases, some one cause of larger error than is unavoidable is brought into the work, and manifests itself by larger discrepancies than the theory of probabilities indicates. In some old series, several such causes are visible: the effect of these is to produce much larger errors. Now-a-days an observer may be called upon to use an old and poor instrument upon distant service, and runs the risk of unusual discrepancies thereby.

So far with regard to the errors of *single* observations : I come now to the smaller errors of series of observations.

If a star's place is determined by *four* observations of equal value, it will be affected with but one-fourth the sum of all the individual errors ; and its probable error, the so-called internal probable error, will be but one-half that of each observation. But there will be new errors introduced, tending to slightly increase this. The skill of the astronomer is shown in making these as small as possible: first, by giving all the parts of his instrument as many reversals as he can, without affecting the stability, reversing his axis end for end, interchanging objective and ocular ; or else by making his observations more strictly differential. The former is best, when fewer objects are to be observed ; the latter, where the mass of work done is more important than its strict independence.

Now I wish to notice that the errors of uniform star places will be more exactly distributed according to the law of probabilities than those of the single observations from which they are formed. For, in the first case, the sources of error are more numerous, and more exactly uniform in their action ; while the resulting errors are more infinitesimal ; provided, that is, due care is taken with the elimination of constant error.

On the other hand, if one star be observed twice, and its neighbor, of just the same importance, twenty times, it will be difficult to bring the probable errors of the results under any general rule.

The *average* rule in first-class observatories is about this : —

Stars observed *en masse*, or by zones, should be twice observed, three times if the two observations disagree much.

Stars observed for ordinary catalogues of objects, interesting as bright or having proper motions, should be *four* times observed, twice in each of two opposite positions of the instrument. Stars requiring special accuracy, not of the very first order, should be observed eight or ten times. For fundamental determinations, it is not so much the number of single observations that is important, as the number of separate determinations (in different years) of eight or ten observations apiece ; and also the number of variations in the position of the instrument and its parts.

For some months I have been bringing together all available material for a catalogue of latitude stars for the United States Engineers. Talcott's method puts a heavy strain on the catalogue, as it is very simple, and easily made accurate with a good instrument, but employs quite small and ill-known stars, for want of well-proportioned pairs enough at any given time and place.

Hence the British Association Catalogue, which contains stars enough, was found almost from the day of its publication to be far from precise. It could hardly be used with a two-inch zenith telescope in indifferent condition, on the boundary between this country and Mexico.

In supplying star-places for this purpose from time to time, the gradual increase of material has been greatly encouraging. But, on the other hand, there is a troublesome want of uniformity in modern star-catalogues. To say nothing of unreduced observations, it often happens that there is much carelessness in settling the zero-points of the instrument, and the clock- and other corrections ; so that the errors in these are often larger than those of observations. Moreover, the star-places used as fundamental are often less accurate than those obtained by their help ; and the various flexures and errors of division, and the variations in clock-rate, are neglected or ill-determined.

As a result, it becomes very difficult to assign the proper weights to the separate determinations, and to settle upon those which are to be excluded. The simplest rule would be to exclude certain doubtful catalogues altogether ; but unfortunately they are sometimes indispensable, where the star is wanted, and no other authority of the same epoch is at hand.

Great assistance has been obtained by noting all the doubtful cases, and requesting their re-observation at the United States Naval Observatory, or re-observing them myself, when other duties would permit; but this is a matter which requires time. I might here mention that the future progress of sidereal astronomy proper will be greatly furthered by continually watching stars of doubtful proper motion until their motions are decided; and that the next few years will very greatly increase the necessity of so doing.

It is, therefore, very necessary that the system of co-operation among the astronomers of different countries, which has lately begun, should extend much wider than it has, or at least that every observer should strive to regulate his work by the uniform principles which guide the best ones, and also to do that which is most necessary for the general good. Fortunately, the making good observations, and reducing them well, requires chiefly industry, system, and order, and is not very dependent upon the capacity to appreciate the highest flights of mathematical genius.

The one thing needful for good observations is, in a word, discipline.

In computing probable errors, I generally use the *sums* of the discrepancies, not the sums of their squares. The formula is, —

$$\varepsilon = 0.845 \frac{\Sigma e}{\sqrt{m(m-n)}}$$

where Σe denotes the sum of the errors, ε the probable error of a single one, m their number, n the number of unknown quantities.

The little table annexed contains

$$M = \frac{\sqrt{m\,(m-n)}}{0.845}$$

so that

$$\varepsilon = \frac{\Sigma e}{M}$$

with the arguments m and n. I have extended it only so far as I habitually use it; beyond these limits, it is better to calculate by logarithms.

In the calculations above-mentioned, I have used two little devices for shortening the solution by least squares, which are best illustrated by the same example.

The star Piazzi xv. 176, has the following determinations of its declination, reduced to the movable planes of 1875, by Bessel's precession and proper systematic corrections : —

Authority.	Declination.	Ep.	Wt.
Piazzi & Lalande .	14°10′51″1	1798	1½
Taylor	47.7	1835	1
Armagh	46.9	1842	1
Quetelet	44.2	1865	¾

The equations to be solved by least squares will be : —

$$x - 0.77\,y = 9''.1 \quad \text{Wt} = 1\tfrac{1}{2}$$
$$x - 0.40\,y = 5.7 \qquad 1$$
$$x - 0.33\,y = 4.9 \qquad 1$$
$$x - 0.10\,y = 2.2 \qquad \tfrac{3}{4}$$

Taking the mean by weights, we have $x - 0.461\ y = 6''.09$, which is the final determination of $x - 0.461\ y$.

Hence,

$$0.309\,y = -\,3''.01$$
$$0.061\,y = -\,0.39$$
$$0.131\,y = -\,1.19$$
$$0.361\,y = -\,3.89$$

with the same weights as before. Solving these equations by least squares (a very easy process), that is, multiplying each by its coefficient of y and its weight, we get —

$$0.142\,y = -\,1''.39$$
$$0.004\,y = -\,0.02$$
$$0.017\,y = -\,0.16$$
$$0.097\,y = -\,1.05$$

Adding $\quad 0.260\,y = -\,2.62$

$$y = -\,10''.08 \quad \text{hence } x = 6''.09 + 0.461 \times -10.08 = 1''.44$$

since the declination for 1875 was $14°10'42'' + x$, and the proper motions in one hundred years y; I took for y in round numbers — 10″ and brought up the declinations with its help as follows : —

		Wt.
Pi. & Lal. .	14°10′43″4	1½
T.	43. 7	1
Arm. . . .	43. 6	1
Q.	43. 2	¾
	Means 43. 48 Adopted 43. 5	} strictly 43″44, see above.

The other process combines Taylor and Armagh into one place ; thus, —

	Authority.	Epoch.	Declination.	Wt.
(1)	Pi. & Lal.	1798	14°10′51″1	1½
(2)	T. & Arm.	1838.5	47. 3	2
(3)	Q.	1865	44. 2	¾

A theorem of Jacobi's tells us to multiply the result of any combination of the two unknown quantities by the square of its determinant, and add all together, and divide by the sum of the squares of the multipliers ; thus, —

Combination.	P. M. Result for one.	Square of Det. × product of wts.		
(1) (3)	—0″103	$67^2 \times 1\frac{1}{2} \times \frac{3}{4}$	=5051	520.3
(1) (2)	—0. 094	$40.5^2 \times 1\frac{1}{2} \times 2$	=4920	462.5
(2) (3)	—0. 117	$26.5^2 \times 2 \times \frac{3}{4}$	=1053	123.2
			11024	1106.0

final mean — 0″.1003.

Using — 0″.10 as before, and bringing up we get

Pi. & Lal. . . .	43″4
T. & Arm. . . .	43. 65
Q.	43. 2
	14°10′43. 48 as before.

Table of $M = \frac{\sqrt{m\,(m-n)}}{0.8454}$

	$n = 1$	$n = 2$
$m = 3$	2 90	2.05
4	4.10	3.35
5	5.29	4.58
6	6.48	5.79
7	7.67	7.00
8	8.85	8.19
9	10.04	9.39
10	11.22	10.58
11	12.41	11.77
12	13.59	12.96
13	14.77	14.14
14	15.96	15.33
15	17.14	16.52
16	18.33	17.70
17	19.51	18.89
18	20.69	20.07
19	21.87	21.26
20	23.05	22.44